Implicit Euler Time-Discretization of a Class of Lagrangian Systems with Set-Valued Robust Controller

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IMPLICIT EULER TIME-DISCRETIZATION OF A CLASS OF LAGRANGIAN SYSTEMS WITH SET-VALUED ROBUST CONTROLLER

SAMIR ADLY*, BERNARD BROGLIATO†, AND BA KHIET LE‡

Abstract. The following class of Lagrangian systems with set-valued controller and subjected to a perturbation force

$$M(q(t))\ddot{q}(t) + C(q(t),\dot{q}(t))\dot{q}(t) + \nabla\mathcal{V}(q(t)) + F(t,q(t),\dot{q}(t)) \in -\partial\Phi(\dot{q}(t)) \;\; a.e.\; t \geq 0, \tag{0.1}$$

has been thoroughly studied in [4]. In this paper, we propose an implicit Euler time-discretized scheme for (0.1)

$$\begin{cases} M(Q^k)\frac{\dot{Q}^{k+1}-\dot{Q}^k}{h_k} + C(Q^k,\dot{Q}^k)\dot{Q}^{k+1} + \nabla\mathcal{V}(Q^k) + F(t_k,Q^k,\dot{Q}^k) \in -\partial\Phi(\dot{Q}^{k+1}) \\ Q^{k+1} = Q^k + h_k\dot{Q}^k. \end{cases} \tag{0.2}$$

Under some mild conditions, the well-posedness (existence and uniqueness of solutions) of the scheme (0.2), as well as the convergence of the sequences $(Q^k), (\dot{Q}^k)$ in finite steps are assured. Furthermore, the approximate piecewise linear function generated by the sequences of points $(Q^k), (\dot{Q}^k)$ converges to the solution of (0.1) with order $\frac{1}{2}$.

Key words. Lagrangian systems, set-valued systems, convergence in finite steps, implicit Euler time-discretization, set-valued analysis

AMS subject classifications. 37N40, 45M10, 46N10, 49M25

1. Introduction.

The dynamics of many systems in physics, mechanics [5], electrical circuits [16, 21], can be formulated by Lagrangian equations. One of the advantages of the method is that it allows one to obtain the same form in any systems of generalized coordinates. Note that the lagrangian function of a system is not unique even though solving any equivalent lagrangian function yields the same equations of motion. The lagrangian $\mathcal{L}(q,\dot{q})$ is defined by the difference between the kinetic energy $\mathcal{T}(q,\dot{q})$ and the potential energy $\mathcal{V}(q)$

$$\mathcal{L}(q,\dot{q}) = \mathcal{T}(q,\dot{q}) - \mathcal{V}(q),$$

where the kinetic energy is usually given by the quadratic form

$$\mathcal{T}(q,\dot{q}) = \frac{1}{2}\langle M(q)\dot{q},\dot{q}\rangle.$$

The matrix $M(q) \in \mathbb{R}^{n\times n}$ is called the inertia matrix, which is positive definite and $M(\cdot)$ is analytic with respect to q in general. With the generalized coordinates $q \in \mathbb{R}^n$, an external force $f \in \mathbb{R}^n$ and a perturbation $F(\cdot,q,\dot{q})$, the Lagrange equations have the form

$$\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{q}}\Big) - \frac{\partial\mathcal{L}}{\partial q} + F(\cdot,q,\dot{q}) = f. \tag{1.1}$$

Using the first kind of Christoffel symbols [15, 19, 22], we can rewrite (1.1) in the form

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla\mathcal{V}(q) + F(\cdot,q,\dot{q}) = f, \tag{1.2}$$

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where $C(q,\dot q)$ is called the centrifugal and Coriolis (and/or moments) matrix which consists of the terms of centrifugal and Coriolis effects satisfying the relationship

$$\frac{d}{dt}\big(M(q(t)\big) = C(q(t),\dot q(t)) + C(q(t),\dot q(t))^T. \tag{1.3}$$

Indeed, the (j,k)-entry of the matrix $C(q,\dot q)$ is given by

$$C_{jk}(q,\dot q) = \sum_{i=1}^{n} C_{ijk}(q)\dot q_i, \tag{1.4}$$

where C_{ijk} is defined by the Christoffel symbols of first kind

$$C_{ijk} = \frac{1}{2}\Big(\frac{\partial M_{jk}}{\partial q_i} + \frac{\partial M_{ji}}{\partial q_k} - \frac{\partial M_{ik}}{\partial q_j}\Big). \tag{1.5}$$

From (1.4), we can write the matrix $C(q,\dot q)$ in the form $C(q,\dot q)=\sum_{i=1}^{n}\dot q_i C_i(q)$, where $C_i(q)$ has the entries $C_{ijk}(q)'s$ and satisfies the equality $C_i(q)+C_i^T(q) = \frac{\partial M}{\partial q_i}(q)$, $i=1,2,\dots$ Therefore

$$C(q,\dot q) + C(q,\dot q)^T = \sum_{i=1}^{n}\dot q_i \frac{\partial M}{\partial q_i} = \frac{d}{dt}\big(M(q)\big). \tag{1.6}$$

Let us emphasize that the centrifugal and Coriolis matrix can be computed with only the knowledge of the inertia matrix, and the equation (1.3) holds for any differentiable function $q(\cdot)$.

In [4] the well-posedness and stability properties of set-valued Lagrangian systems like

$$M(q(t))\ddot q(t) + C(q(t),\dot q(t))\dot q(t) + \nabla\mathcal{V}(q(t)) + F(t,q(t),\dot q(t)) \in -\partial\Phi(\dot q(t)) \;\; a.e.\ t\ge t_0,$$

where $\Phi(\cdot)$ is a scalar convex function, has been analyzed. The terms $\nabla\mathcal{V}(q(t))$ and $\partial\Phi(\dot q(t))$ may represent a control input $u(q,\dot q) = -\nabla\mathcal{V}(q) - \partial\Phi(\dot q)$ applied to stabilize the system at some fixed point. This is a general framework for robust control, with a quite important particular case represented by sliding mode control [18, 20, 23, 24]. The example treated in section 6 illustrates this point. Time discretization of set-valued, sliding mode controllers is crucial for practical implementations of such control techniques on computers, virtual prototyping, and chattering attenuation analysis [1, 2]. Therefore, in this paper, we propose the following implicit time-discretized scheme of (0.1) and analyze its properties

$$\begin{cases} M(Q^k)\frac{\dot Q^{k+1}-\dot Q^k}{h_k} + C(Q^k,\dot Q^k)\dot Q^{k+1} + \nabla\mathcal{V}(Q^k) + F(t_k,Q^k,\dot Q^k) \in -\partial\Phi(\dot Q^{k+1}) \\ Q^{k+1} = Q^k + h_k\dot Q^k, \end{cases} \tag{1.7}$$

where (h_k) is a sequence of positive steps. It is noteworthy that the set-valued term in (1.7) is discretized in an implicit way.

The paper is organized as follows. In Section 2, we recall certain conventional notations and prerequisites which are used throughout the paper. Then some crucial properties and assumptions are proposed in Section 3. In Section 4, firstly the existence and uniqueness of (Q^k), $(\dot Q^k)$ satisfying the system (0.2) is proved. Then, under the assumption on the boundedness, time independence of the perturbation force $F(\cdot,\cdot,\cdot)$, the boundedness of $\nabla\mathcal{V}(\cdot)$ and the domination of $\Phi(\cdot)$ over a quadratic form of norm function, the approximate velocity sequence $(\dot{Q}^k)$ is shown to vanish asymptotically and the approximate position (Q^k) converges to Q^∞, where Q^∞ satisfies the relation: $0 \in \partial\Phi(0) + F(Q^\infty, 0) + \nabla\mathcal{V}(Q^\infty)$, *i.e.*, Q^∞ is an equilibrium of (0.1) and (0.2). In addition, under the condition: $-F(Q^\infty, 0) - \nabla\mathcal{V}(Q^\infty) \in \mathrm{int}(\partial\Phi(0))$, the sequence (Q^k) is proved to converge in a finite number of steps. This kind of condition is often used in literature to obtain the finite time convergence property of the continuous and discrete systems [4, 6]. In Section 5, we use a sequence of piecewise linear functions to approximate the solutions of the continuous systems and it is proved that the convergence order is $\frac{1}{2}$. Section 6 is dedicated to numerical simulations on a non linear two-degree-of-freedom fully actuated Lagragian system. Conclusions end the paper in Section 7.

2. Notations and Mathematical Background. We first introduce some notations used in the paper. Denote by $\langle\cdot,\cdot\rangle$, $\|\cdot\|$ the Euclidean scalar product and the corresponding norm in $\mathbb{R}^n$; by $\|\cdot\|_m$ the induced matrix norm defined as follows

$$\|M\|_m = \sup_{\|x\|=1, x\in\mathbb{R}^n} \|Mx\|, \tag{2.1}$$

where M is a square matrix of size n. The supremum norm, the L^∞-norm, the L^2-norm of a given function from $[0,T]$ to $\mathbb{R}^n$ are denoted by $\|\cdot\|_{\mathcal{C}(0,T;\mathbb{R}^n)}$, $\|\cdot\|_{L^\infty(0,T;\mathbb{R}^n)}$, $\|\cdot\|_{L^2(0,T;\mathbb{R}^n)}$ respectively. Denote by I the identity operator, I_n the identity matrix of size n; $\mathbb{B}_\varepsilon$ the closed ball of radius ε centered at 0 and $\mathbb{B}_\varepsilon(x)$ the closed ball of radius ε centered at x. The interior and boundary of a given set $S \in \mathbb{R}^n$ are denoted by $\mathrm{int}(S)$ and $\mathrm{bd}(S)$ respectively. Suppose that S is a nonempty, closed and convex set. The normal cone to S at the point x in S is given by

$$N_S(x) = \{p \in \mathbb{R}^n : \langle p, y - x\rangle \le 0 \text{ for all } y \in S\}.$$

The projection of a point x on the set S is the unique point belonging to S which is closest to x

$$\mathrm{proj}[S; x] = y \text{ such that } \|y - x\| = \min_{z\in S} \|z - x\|.$$

Let us recall from convex analysis that the subdifferential of the indicator function at $x \in S$ is the normal cone to S at x

$$\partial\Psi_S(x) = N_S(x). \tag{2.2}$$

In addition, let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function, then we have the following relation

$$x^* \in \partial f(x) \Leftrightarrow x \in \partial f^*(x^*) \tag{2.3}$$

where $f^*(\cdot)$ is the conjugate function of $f(\cdot)$. Let $A(\cdot)$ be a set-valued map from $\mathbb{R}^n$ into the subsets of $\mathbb{R}^n$. The domain of $A(\cdot)$ is defined by $\mathrm{dom}(A) = \{x \in \mathbb{R}^n : A(x) \neq \emptyset\}$. $A(\cdot)$ is said *monotone* provided

$$\langle x_2 - x_1, y_2 - y_1\rangle \ge 0 \quad \text{for all } x_1, x_2 \in \mathbb{R}^n \text{ and } y_1 \in A(x_1), y_2 \in A(x_2).$$

The monotone map $A(\cdot)$ is called *maximal monotone* if there is no other monotone set-valued map $B(\cdot)$ such that the graph $G(A) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : y \in A(x)\}$ is contained strictly in the graph of $B(\cdot)$. Given a maximal monotone map $A(\cdot)$ and a positive real number λ, the *resolvent of* A (of index λ) defined by $J_\lambda^A := (I + \lambda A)^{-1}$ is a non-expansive single valued map from $\mathbb{R}^n$ to $\mathbb{R}^n$. Denote by $m(A(x))$ the set of elements of $A(x)$ with minimal norm, i.e.,

$$m(A(x)) := \{y^* \in A(x) : \|y^*\| = \min_{y \in A(x)} \|y\|\} \tag{2.4}$$

and

$$\|m(A(x))\| := \min_{y \in A(x)} \|y\|. \tag{2.5}$$

If $A(\cdot)$ is a linear single-valued mapping then $A(\cdot)$ is maximal monotone if and only if

$$\langle Ax, x \rangle \geq 0, \quad \text{for all } x \in \mathbb{R}^n. \tag{2.6}$$

The following result provides a sufficient criterion for verifying whether the sum of two maximal monotone operators is also maximal monotone. Let $A(\cdot)$ and $B(\cdot)$ be maximal monotone operators on $\mathbb{R}^n$ and $\mathrm{dom}(A) \cap \mathrm{int}(\mathrm{dom}(B)) \neq \emptyset$, then $A(\cdot) + B(\cdot)$ is maximal monotone. Particularly, it is also true if $B : \mathbb{R}^n \to \mathbb{R}^n$ is single-valued, monotone and continuous, see [8].

3. Properties and Assumptions. Let us propose some important assumptions that are used in the analysis led in the sequel. Note that the assumptions 3.1, 3.2 may be considered as crucial properties of Lagrangian systems [12, 19, 22], for example the assumption 3.2 ($\mathcal{C}_2$), see (1.3) and the relevant comment in the introduction, is essential for the stability and the control of Lagrangian systems [15].

Assumption 3.1 (The inertia matrix).
($\mathcal{M}_1$) *For all* $q \in \mathbb{R}^n$, $M(q)$ *is symmetric and there exist* $k_1 > 0, k_2 > 0$ *such that*

$$k_1 I_n \leq M(q) \leq k_2 I_n, \tag{3.1}$$

i.e. for all $q, x \in \mathbb{R}^n$, *we have:* $k_1\|x\|^2 \leq \langle M(q)x, x\rangle \leq k_2\|x\|^2$.

($\mathcal{M}_2$) $M(\cdot) = (m_{ij}(\cdot))_{1 \leq i,j \leq n} \in \mathcal{C}^2(\mathbb{R}^n; \mathbb{R}^{n \times n})$, *i.e.,* $m_{ij}(\cdot) \in \mathcal{C}^2(\mathbb{R}^n; \mathbb{R})$.

Assumption 3.2 (The centrifugal and Coriolis matrix).

($\mathcal{C}_1$) *There exists* $k_3 > 0$ *such that for all* $u, v \in \mathbb{R}^n$ *:*

$$\|C(u, v)\|_m \leq k_3\|v\|.$$

($\mathcal{C}_2$) *Given any differentiable function* $u : \mathbb{R}_+ \to \mathbb{R}^n$, *we have for all* $t \geq t_0$*:*

$$\frac{d}{dt}(M(u(t))) = C(u(t), \dot{u}(t)) + C(u(t), \dot{u}(t))^T.$$

($\mathcal{C}_3$) *The function* $h : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *defined by* $h(x_1, x_2, x_3) = C(x_1, x_2)x_3$ *is locally Lipschitz.*

Assumption 3.3.

($\mathcal{H}_F$) *The function* $F(t,x_1,x_2)$ *from* $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ *is continuous in* t*, uniformly locally Lipschitz in* x_1,x_2 *(i.e. the Lipschitz constant is independent of* t*).*

($\mathcal{H}_\Phi$) $\Phi(\cdot)$ *is a convex function on* $\mathbb{R}^n$*.*

($\mathcal{H}_\mathcal{V}$) $\mathcal{V}(\cdot)$ *is a differentiable function on* $\mathbb{R}^n$ *and* $\nabla\mathcal{V}$ *is locally Lipschitz.*

The Assumption 3.1 ($\mathcal{M}_2$) implies the Lipschitzness on bounded sets of the inertia matrix $M(\cdot)$. Indeed, we have

LEMMA 3.1. *Given* $K>0$*, then there exists* $k_4>0$ *such that for all* $q_1,q_2,u\in\mathbb{R}^n$ *and* $\|q_1\|,\|q_2\|\le K$*:*

$$\|M(q_1)u-M(q_2)u\|\le k_4\|q_1-q_2\|\|u\|. \tag{3.2}$$

Proof. Fix $u\in\mathbb{R}^n$. Let the function $G:\mathbb{R}^n\to\mathbb{R}^n$ be defined by

$$G(x):=M(x)u. \tag{3.3}$$

Then $G(\cdot)$ is a twice differentiable function using ($\mathcal{M}_2$). We have

$$G(x)=M(x)u=\begin{pmatrix}\sum_{j=1}^n m_{1j}(x)u_j\\ \dots\\ \sum_{j=1}^n m_{nj}(x)u_j\end{pmatrix}=\sum_{j=1}^n\begin{pmatrix}m_{1j}(x)\\ \dots\\ m_{nj}(x)\end{pmatrix}u_j, \tag{3.4}$$

and its Jacobian matrix

$$J_G(x)=\sum_{j=1}^n\begin{pmatrix}\nabla m_{1j}(x)^T\\ \dots\\ \nabla m_{nj}(x)^T\end{pmatrix}u_j. \tag{3.5}$$

Then using the Mean Value Theorem, the conclusion follows. Note that the constant k_4 depends only on K and the matrix $M(q)$. □

The following lemmas show that the inverse matrix $M^{-1}(\cdot)$ is also bounded and Lipschitz on bounded sets. Their proofs are analog to the proofs in Lemma 3.3 and Lemma 3.4 in [4].

LEMMA 3.2. *For all* $q\in\mathbb{R}^n$*,* $M^{-1}(q)$ *is symmetric and there exist* $k_5>0,k_6>0$ *such that*

$$k_5I_n\le M^{-1}(q)\le k_6I_n. \tag{3.6}$$

LEMMA 3.3. *Given* $K>0$*, then there exists* $k_7>0$ *such that for all* $q_1,q_2,u\in\mathbb{R}^n$ *and* $\|q_1\|,\|q_2\|\le K$*:*

$$\|M^{-1}(q_1)u-M^{-1}(q_2)u\|\le k_7\|q_1-q_2\|\|u\|. \tag{3.7}$$

4. Well-posedness and Asymptotic Behavior Analysis. Let us begin with the existence and uniqueness results of the sequences (Q^k), $(\dot{Q}^k)$ defined in (0.2). Then under some mild conditions, the convergence of (Q^k), $(\dot{Q}^k)$ is obtained, even in finite steps. The following theorem assures the well-posedness of the scheme.

THEOREM 4.1. *Let Assumptions* 3.1, 3.2 $(\mathcal{C}_1)$ *hold. Consider the implicit scheme in (0.2)*

$$\begin{cases} M(Q^k)\frac{\dot{Q}^{k+1}-\dot{Q}^k}{h_k} + C(Q^k,\dot{Q}^k)\dot{Q}^{k+1} + \nabla\mathcal{V}(Q^k) + F(t_k,Q^k,\dot{Q}^k) \in -\partial\Phi(\dot{Q}^{k+1}) \\ \\ Q^{k+1} = Q^k + h_k\dot{Q}^k. \end{cases}$$

Fix some $h > 0$. Consider the sequences $(Q^k), (\dot{Q}^k)$ generated by the following algorithm:

1. *$k := 0, t_0 := 0$ and take the starting points $Q^0, \dot{Q}^0 \in \mathbb{R}^n$.*
2. *Given $(t_k, Q^k, \dot{Q}^k)$, we choose $h_k := \min\{h, \frac{k_1}{2k_3\|\dot{Q}^k\|}\}$, where k_1, k_3 are defined in the assumptions* 3.1, 3.2 *and find $(Q^{k+1}, \dot{Q}^{k+1})$ satisfying* (0.2). *Let $t_{k+1} := t_k + h_k$.*
3. *Let $k := k+1$ and go to step 2.*

Then at each step k, the approximated position Q^{k+1} and approximated velocity $\dot{Q}^{k+1}$ can be uniquely calculated from the knowledge of Q^k, $\dot{Q}^k, t_k$ and h_k.

Proof. Fix some positive integer k. Obviously, we can compute Q^{k+1} directly by the formula $Q^{k+1} = Q^k + h_k\dot{Q}^k$. From the assumption 3.2 $(\mathcal{C}_1)$ we have

$$\|C(Q^k,\dot{Q}^k)\|_m \le k_3\|\dot{Q}^k\|. \tag{4.1}$$

Since $h_k = \min\{h, \frac{k_1}{2k_3\|\dot{Q}^k\|}\} \le \frac{k_1}{2k_3\|\dot{Q}^k\|}$, it follows

$$h_k k_3\|\dot{Q}^k\| \le \frac{k_1}{2} < k_1. \tag{4.2}$$

Let $B^k(Q^k,\dot{Q}^k) := \frac{1}{h_k}M(Q^k)+C(Q^k,\dot{Q}^k)-\frac{k_1}{2h_k}I_n = \frac{1}{h_k}\{M(Q^k)-k_1I_n\}+\{C(Q^k,\dot{Q}^k)+\frac{k_1}{2h_k}I_n\}$ and $\lambda_k := \frac{k_1}{2h_k} > 0$. Then $B^k(\cdot)$ is a linear single-valued mapping and for all $x \in \mathbb{R}^n$, we have

$$\begin{aligned} \langle B^k x, x\rangle &= \tfrac{1}{h_k}\langle\{M(Q^k)-k_1I_n\}x,x\rangle + \tfrac{1}{h_k}\langle\{h_kC(Q^k,\dot{Q}^k)+\tfrac{k_1}{2}I_n\}x,x\rangle \\ &\ge \tfrac{1}{h_k}(\tfrac{k_1}{2} - h_kk_3\|\dot{Q}^k\|)\|x\|^2 \ge 0, \end{aligned} \tag{4.3}$$

using (4.2) and Assumption 3.1. Hence $B^k(\cdot)$ is maximal monotone with $\mathrm{dom}(B^k) = \mathbb{R}^n$. Note that $\partial\Phi(\cdot)$ is maximal monotone since $\Phi(\cdot)$ is a convex function on $\mathbb{R}^n$, see [8]. The set-valued mapping $A^k := B^k + \partial\Phi$ is therefore also maximal monotone. After some simple computations, we obtain

$$M(Q^k)\frac{\dot{Q}^k}{h_k} - F(t_k,Q^k,\dot{Q}^k) - \nabla\mathcal{V}(Q^k) \in (A^k+\lambda_kI)(\dot{Q}^{k+1}),$$

which implies that

$$\dot{Q}^{k+1} = J^{A^k}_{\lambda_k}\{M(Q^k)\frac{\dot{Q}^k}{h_k} - F(t_k,Q^k,\dot{Q}^k) - \nabla\mathcal{V}(Q^k)\},$$

where $J_{\lambda_k}^{A_k} = (A^k + \lambda_k I)^{-1}$ is the resolvent of the maximal monotone mapping $A_k(\cdot)$ of index λ_k. In conclusion, we can compute $\dot{Q}^{k+1}$, Q^{k+1} in terms of Q^k, $\dot{Q}^k$ and h_k uniquely. ◻

REMARK 4.1. *The choice of* $h_k = \min\{h, \frac{k_1}{2k_3\|\dot{Q}^k\|}\} \le h$ *is given in order to avoid large values of* h_k *when* $\dot{Q}^k$ *is very close to zero.*

In the following part, we analyze the convergence of (Q^k), $(\dot{Q}^k)$ based on the existence of a Lyapunov's sequence which is non-increasing along the discrete trajectories. Let us begin with the following assumption:

ASSUMPTION 4.1. *There exist* $\alpha > \beta \ge 0$ *and* $\eta, \xi > 0$ *such that*

1) $\Phi(\cdot) \ge \eta\|\cdot\|^2 + \alpha\|\cdot\| + \Phi(0)$.

2) $\sup_{(t,x_1,x_2)\in\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n} \|F(t,x_1,x_2)\| \le \beta$.

3) $\|\nabla\mathcal{V}(x)\| \le \frac{\alpha-\beta}{2}$ *for all* $x \in \mathbb{R}^n$.

REMARK 4.2. *If from the collected data, we can estimate an upper bound of the continuous perturbation force* $F(\cdot)$ *by some positive* β*, then we may choose* $\Phi(\cdot) := \eta\|\cdot\|^2 + \alpha\|\cdot\|$ *for some positive* η, α *with* $\alpha > \beta$ *and choose* $\mathcal{V}(\cdot)$ *by some constant functions, or linear functions, or* $\mathcal{V}(\cdot) := \frac{\alpha-\beta}{2}\|\cdot\|_\lambda$ *where* $\|\cdot\|_\lambda$ *is the Moreau-Yosida approximation of the Euclidean norm function of index* λ *for some* $\lambda > 0$*. Then Assumptions* 3.3, 4.1 *are satisfied. Indeed, for each* λ*, it is known that* $\frac{\alpha-\beta}{2}\|\cdot\|_\lambda$ *is differentiable and* $\|\nabla\frac{\alpha-\beta}{2}\|\cdot\|_\lambda\| \le \frac{\alpha-\beta}{2}\|m(\partial\|x\|)\| \le \frac{\alpha-\beta}{2}$.

Let us now investigate how property (1.3), which is instrumental for the analysis led in [4] and more generally for the stability analysis of controlled Lagrangian systems [15], may be transposed in the discrete time context. Given $(Q^k, \dot{Q}^k)$, we consider the following linear function $u : \mathbb{R}^+ \to \mathbb{R}^n$ defined by

$$u(t) = Q^k + \dot{Q}^k(t - t_0), \tag{4.4}$$

for some $t_0 > 0$. From Assumption 3.2 $(\mathcal{C}_2)$, we obtain

$$\frac{d}{dt}M(u(t)) = C(u(t), \dot{u}(t)) + C(u(t), \dot{u}(t))^T = C(u(t), \dot{Q}^k) + C(u(t), \dot{Q}^k)^T \tag{4.5}$$

for all $t \ge 0$. Take $t = t_0$ in (4.5), then

$$\frac{d}{dt}M(u(t_0)) = C(Q^k, \dot{Q}^k) + C(Q^k, \dot{Q}^k)^T. \tag{4.6}$$

On the other hand, by using Taylor's expansion and Assumption 3.1 $(\mathcal{M}_2)$, we have

$$M(u(t_0 + h_k)) = M(u(t_0)) + h_k\frac{d}{dt}M(u(t_0)) + \mathcal{O}(h_k^2) \tag{4.7}$$

or equivalently,

$$\frac{M(Q^k + h_k\dot{Q}^k) - M(Q^k)}{h_k} = C(Q^k, \dot{Q}^k) + C(Q^k, \dot{Q}^k)^T + \mathcal{O}(h_k). \tag{4.8}$$

Hence, we can find $h_k^* > 0$ such that for all $h_k \le h_k^*$ then

$$\varepsilon_k = \varepsilon_k(h_k, Q^k, \dot{Q}^k) := \frac{M(Q^k + h_k\dot{Q}^k) - M(Q^k)}{h_k} - C(Q^k, \dot{Q}^k) - C(Q^k, \dot{Q}^k)^T \tag{4.9}$$

satisfying $\|\varepsilon_k\|_m \le 2\eta$, where $\eta > 0$ is defined in Assumption 4.1. Note that h_k^* depends only on η, M, Q^k and $\dot{Q}^k$. Thus we have the following theorem:

THEOREM 4.2. *Let Assumptions* 3.1, 3.2, 3.3, 4.1 *hold and consider the modified algorithm:*

Fix some $h > 0$. Consider the sequences $(Q^k), (\dot{Q}^k)$ generated by

1. *$k := 0, t_0 = 0$ and take the initial data $Q^0, \dot{Q}^0 \in \mathbb{R}^n, h_{-1} = h$.*
2. *Given $(Q^k, \dot{Q}^k)$, we choose $h_k := \min\{h_{k-1}, h_k^*, \frac{k_1}{2k_3\|\dot{Q}^k\|}\}$ where h_k^* is defined in* (4.9)*; k_1, k_3 are defined in Assumptions* 3.1, 3.2. *Find $(Q^{k+1}, \dot{Q}^{k+1})$ satisfying* (0.2). *Let $t_{k+1} := h_k + t_k$.*
3. *Let $k := k+1$ and go to step 2.*

Then the sequence (Q^k) converges to a point denoted Q^∞ , $\lim_{k\to\infty} \dot{Q}^k = 0$ and $\lim_{k\to\infty} t_k = \infty$.

Proof. From Theorem 4.1, for each $k > 0$ we have that Q^{k+1} and $\dot{Q}^{k+1}$ can be uniquely computed in terms of Q^k, $\dot{Q}^k$ and h_k. So the sequences $(Q^k), (\dot{Q}^k)$ are well-defined. Furthermore, from (4.9) and the choice of h_k, we have

$$\begin{aligned} \varepsilon_k &= \tfrac{M(Q^k + h_k\dot{Q}^k) - M(Q^k)}{h_k} - C(Q^k, \dot{Q}^k) - C(Q^k, \dot{Q}^k)^T \\ &= \tfrac{M(Q^{k+1}) - M(Q^k)}{h_k} - C(Q^k, \dot{Q}^k) - C(Q^k, \dot{Q}^k)^T \end{aligned} \tag{4.10}$$

and $\|\varepsilon_k\|_m \le 2\eta$. Let us define the Lyapunov's sequence (E_k) by

$$E_k = \frac{1}{2}\langle M(Q^k)\dot{Q}^k, \dot{Q}^k\rangle$$

for each positive integer k. Let us prove that (E_k) is non-increasing. In fact, we have

$$E_{k+1} - E_k = \frac{1}{2}\langle M(Q^{k+1})\dot{Q}^{k+1}, \dot{Q}^{k+1}\rangle - \frac{1}{2}\langle M(Q^k)\dot{Q}^k, \dot{Q}^k\rangle.$$

Note that

$$\begin{aligned} &\langle M(Q^{k+1})\dot{Q}^{k+1}, \dot{Q}^{k+1}\rangle - \langle M(Q^k)\dot{Q}^k, \dot{Q}^k\rangle \\ &= \langle (M(Q^{k+1}) - M(Q^k))\dot{Q}^{k+1}, \dot{Q}^{k+1}\rangle + \langle M(Q^k)(\dot{Q}^{k+1} - \dot{Q}^k), \dot{Q}^{k+1}\rangle + \langle M(Q^k)\dot{Q}^k, \dot{Q}^{k+1} - \dot{Q}^k\rangle \\ &= \langle (M(Q^{k+1}) - M(Q^k))\dot{Q}^{k+1}, \dot{Q}^{k+1}\rangle + 2\langle M(Q^k)(\dot{Q}^{k+1} - \dot{Q}^k), \dot{Q}^{k+1}\rangle \\ &\quad - \langle M(Q^k)(\dot{Q}^{k+1} - \dot{Q}^k), \dot{Q}^{k+1} - \dot{Q}^k\rangle \\ &\le \langle (M(Q^{k+1}) - M(Q^k))\dot{Q}^{k+1}, \dot{Q}^{k+1}\rangle + 2\langle M(Q^k)(\dot{Q}^{k+1} - \dot{Q}^k), \dot{Q}^{k+1}\rangle \\ &\le 2h_k\big(\langle M(Q^k)\frac{(\dot{Q}^{k+1} - \dot{Q}^k)}{h_k} + C(Q^k, \dot{Q}^k)\dot{Q}^{k+1}, \dot{Q}^{k+1}\rangle + \frac{\|\varepsilon_k\|_m}{2}\|\dot{Q}^{k+1}\|^2\big) \\ &\le 2h_k\big(\langle M(Q^k)\frac{(\dot{Q}^{k+1} - \dot{Q}^k)}{h_k} + C(Q^k, \dot{Q}^k)\dot{Q}^{k+1}, \dot{Q}^{k+1}\rangle + \eta\|\dot{Q}^{k+1}\|^2\big) \end{aligned}$$

where we used Assumption 3.1 and (4.10). Let $r_{k+1} \in \partial\Phi(\dot{Q}^{k+1})$ be such that

$$M(Q^k)\frac{\dot{Q}^{k+1} - \dot{Q}^k}{h_k} + C(Q^k, \dot{Q}^k)\dot{Q}^{k+1} + \nabla\mathcal{V}(Q^k) + F(t_k, Q^k, \dot{Q}^k) = -r_{k+1}. \tag{4.11}$$

Therefore

$$\begin{aligned} \frac{E_{k+1} - E_k}{h_k} &\le \langle M(Q^k)\frac{(\dot{Q}^{k+1} - \dot{Q}^k)}{h_k} + C(Q^k, \dot{Q}^k)\dot{Q}^{k+1}, \dot{Q}^{k+1}\rangle + \eta\|\dot{Q}^{k+1}\|^2 \\ &= -\langle \nabla\mathcal{V}(Q^k) + F(t_k, Q^k, \dot{Q}^k) + r_{k+1}, \dot{Q}^{k+1}\rangle + \eta\|\dot{Q}^{k+1}\|^2. \end{aligned}$$

From the definition of the subdifferential, we obtain that

$$\begin{aligned}\frac{E_{k+1}-E_k}{h_k} &\le -\langle \nabla \mathcal{V}(Q^k) + F(t_k, Q^k, \dot{Q}^k), \dot{Q}^{k+1}\rangle - \Phi(\dot{Q}^{k+1}) + \Phi(0) + \eta\|\dot{Q}^{k+1}\|^2 \\ &\le \frac{\alpha-\beta}{2}\|\dot{Q}^{k+1}\| + \beta\|\dot{Q}^{k+1}\| - \alpha\|\dot{Q}^{k+1}\| \le -\frac{\alpha-\beta}{2}\|\dot{Q}^{k+1}\|, \qquad (4.12)\end{aligned}$$

due to Assumption 4.1. Hence

$$E_{k+1} - E_k \le -h_k\frac{\alpha-\beta}{2}\|\dot{Q}^{k+1}\| \le -h_{k+1}\frac{\alpha-\beta}{2}\|\dot{Q}^{k+1}\| = -\frac{\alpha-\beta}{2}\|Q^{k+2}-Q^{k+1}\| \qquad (4.13)$$

since $h_{k+1} \le h_k$, and thus

$$\|Q^{k+2}-Q^{k+1}\| \le \frac{2}{\alpha-\beta}(E_k - E_{k+1}). \qquad (4.14)$$

Then take the sum side by side of (4.14) from $k=0$ to N for any positive integer N and let $N \to +\infty$. We obtain that $(\|Q^{k+1}-Q^k\|) \in l^1$ because (E_k) is bounded from below by zero. It means that (Q^k) is a Cauchy sequence and hence the convergence of (Q^k) follows due to the completeness of $\mathbb{R}^n$. Let $Q^\infty = \lim_{k\to\infty} Q^k$. In particular, (Q^k) is bounded. On the other hand, from (4.13), the sequence (E_k) is non-increasing. Therefore

$$\frac{1}{2}k_1\|\dot{Q}^k\|^2 \le \frac{1}{2}\langle M(Q^k)\dot{Q}^k, \dot{Q}^k\rangle = E_k \le E_0, \qquad (4.15)$$

due to Assumptions 3.1. It leads to the boundedness of $(\dot{Q}^k)$. It is obvious that (h_k) is bounded from above by h. Now we prove that (h_k) is also bounded from below. Indeed, the boundedness of $(Q^k), (\dot{Q}^k)$ implies the boundedness of (h_k^*) (see (4.9) for the definition of h_k^*). Hence $h_k = \min\{h_{k-1}, h_k^*, \frac{k_1}{2k_3\|\dot{Q}^k\|}\}$ is bounded from below. The sequence (h_k) is non-increasing by construction and bounded from below, so there exists a $\bar{h} > 0$ such that $\lim_{k\to\infty} h_k = \bar{h}$. Hence for each $k > 0$, $h_k \ge \bar{h}$ and $t_{k+1} = t_k + h_k \ge t_k + \bar{h} \ge t_0 + (k+1)\bar{h}$ which implies the limit $\lim_{k\to\infty} t_k = +\infty$. From the formula $\dot{Q}^k = \frac{Q^{k+1}-Q^k}{h_k}$, one infers that $\lim_{k\to\infty}\dot{Q}^k = 0$. □

THEOREM 4.3. *Suppose that the perturbation force is independent of time, i.e.*

$$F(t, x_1, x_2) \equiv F(x_1, x_2), \ (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n. \qquad (4.16)$$

Let the assumptions of Theorem 4.2 hold. Then the sequence (Q^k) generated by the algorithm in Theorem 4.2 converges to an equilibrium of (0.2), *i.e.,* $F(Q^\infty, 0) + \nabla\mathcal{V}(Q^\infty) \in -\partial\Phi(0)$. *Furthermore, if* $-F(Q^\infty, 0) - \nabla\mathcal{V}(Q^\infty) \in \mathrm{int}(\partial\Phi(0))$ *then the sequence (Q^k) is finitely convergent, i.e.,* $\exists\, N \in \mathbb{Z}_+$ *such that* $\forall\, k \ge N : Q^k = Q^\infty$.

Proof. For each $k = 1, 2...$, one knows that

$$M(Q^k)\frac{\dot{Q}^{k+1}-\dot{Q}^k}{h_k} + C(Q^k, \dot{Q}^k)\dot{Q}^{k+1} + F(Q^k, \dot{Q}^k) + \nabla\mathcal{V}(Q^k) \in -\partial\Phi(\dot{Q}^{k+1}). \quad (4.17)$$

The Assumptions $3.1, 3.2$ $(\mathcal{C}_1)$ and Theorem 4.2 imply

$$\|M(Q^k)\frac{\dot{Q}^{k+1}-\dot{Q}^k}{h_k} + C(Q^k, \dot{Q}^k)\dot{Q}^{k+1}\| \le k_2\|\frac{\dot{Q}^{k+1}-\dot{Q}^k}{h_k}\| + k_3\|\dot{Q}^k\|\|\dot{Q}^{k+1}\| \to 0 \qquad (4.18)$$

as $k \to +\infty$. Therefore

$$M(Q^k)\frac{\dot{Q}^{k+1}-\dot{Q}^k}{h_k}+C(Q^k,\dot{Q}^k)\dot{Q}^{k+1}+F(Q^k,\dot{Q}^k)+\nabla\mathcal{V}(Q^k)\to F(Q^\infty,0)+\nabla\mathcal{V}(Q^\infty) \tag{4.19}$$

as $k \to +\infty$ since $\nabla\mathcal{V}(\cdot), F(\cdot)$ are continuous. Using the closed graph property of the maximal monotone mapping $\partial\Phi(\cdot)$ (see for example [7]), we obtain: $F(Q^\infty,0)+\nabla\mathcal{V}(Q^\infty)\in-\partial\Phi(0)$. It means that Q^∞ is an equilibrium of the scheme (0.2).

In the next part, we prove that the sequence (Q^k) converges in a finite number of steps under the condition $-F(Q^\infty,0)-\nabla\mathcal{V}(Q^\infty)\in\mathrm{int}(\partial\Phi(0))$. Indeed, let $\mathcal{E}$ be the set $\{k\in\mathbb{N}: Q^{k+1}=Q^k\}$. Suppose that the sequence (Q^k) is not finitely convergent. Let Ω be the set of limit points of $(\frac{Q^{k+1}-Q^k}{\|Q^{k+1}-Q^k\|})_{k\in\mathbb{N}\setminus\mathcal{E}}$ and let $r\in\Omega$. Note that Ω is nonempty since $(\frac{Q^{k+1}-Q^k}{\|Q^{k+1}-Q^k\|})_{k\in\mathbb{N}\setminus\mathcal{E}}$ belongs to the unit sphere which is a compact subset of $\mathbb{R}^n$. Hence, there exists a subsequence of (Q^k), still denoted by (Q^k) such that $\lim_{k\to\infty}\frac{Q^{k+1}-Q^k}{\|Q^{k+1}-Q^k\|}=r$ with $\|r\|=1$. Let us choose some $\zeta\in\partial\Phi(0)$ and take $r_{k+1}\in\partial\Phi(\dot{Q}^{k+1})$ defined in (4.11). Using the monotonicity of $\partial\Phi(\cdot)$, we obtain $\langle r_{k+1}-\zeta,\dot{Q}^{k+1}\rangle\geq 0$. Hence, from (0.2), one infers

$$\langle -M(Q^k)\frac{\dot{Q}^{k+1}-\dot{Q}^k}{h_k}-C(Q^k,\dot{Q}^k)\dot{Q}^{k+1}-F(Q^k,\dot{Q}^k)-\nabla\mathcal{V}(Q^k)-\zeta,\frac{Q^{k+2}-Q^{k+1}}{\|Q^{k+2}-Q^{k+1}\|}\rangle\geq 0.$$

By taking the limit when $k\to+\infty$, we have:$\langle -F(Q^\infty,0)-\nabla\mathcal{V}(Q^\infty)-\zeta,r\rangle\geq 0$ for all $\zeta\in\partial\Phi(0)$. Combining with $-F(Q^\infty,0)-\nabla\mathcal{V}(Q^\infty)\in\mathrm{int}(\partial\Phi(0))$, it is trivial to conclude that $r=0$ since we can choose $\zeta\in\partial\Phi(0)$ such that $F(Q^\infty,0)+\nabla\mathcal{V}(Q^\infty)+\zeta=mr$ for some real number $m>0$. It is a contradiction with the fact that $\|r\|=1$. Hence, the sequence (Q^k) converges in finite steps. □

5. Convergence of piecewise linear approximations. We have proved that for each initial data $(Q^0,\dot{Q}^0)$ and for each positive h, one can generate the unique sequences $(Q^k),(\dot{Q}^k),(h_k),(t_k)$ by the algorithm in Theorem 4.2, satisfying

$$\begin{cases} M(Q^k)\frac{\dot{Q}^{k+1}-\dot{Q}^k}{h_k}+C(Q^k,\dot{Q}^k)\dot{Q}^{k+1}+\nabla\mathcal{V}(Q^k)+F(t_k,Q^k,\dot{Q}^k)\in-\partial\Phi(\dot{Q}^{k+1})\\ Q^{k+1}=Q^k+h_k\dot{Q}^k,\end{cases}$$

where $(Q^k),(\dot{Q}^k),(h_k)$ are bounded sequences (particularly (h_k) is bounded above from h) and $\lim_{k\to+\infty}Q^k=Q^\infty,\lim_{k\to+\infty}\dot{Q}^k=0,\lim_{k\to+\infty}t_k=+\infty$. Given $T>0$. Let $q_h(\cdot)$ and $v_h(\cdot)$ be the linear interpolations of the Q^k's and $\dot{Q}^k$'s on $[0,T]$, respectively. It means that for each positive integer k and $t\in[t_k,t_{k+1})$

$$q_h(t)=Q^k+\frac{Q^{k+1}-Q^k}{t_{k+1}-t_k}(t-t_k)=Q^k+\dot{Q}^k(t-t_k) \tag{5.1}$$

$$v_h(t)=\dot{Q}^k+\frac{\dot{Q}^{k+1}-\dot{Q}^k}{t_{k+1}-t_k}(t-t_k)=\dot{Q}^k+\frac{\dot{Q}^{k+1}-\dot{Q}^k}{h_k}(t-t_k). \tag{5.2}$$

We define also the following step functions on $[0, T]$

$$q_{*h}(t) = Q^k \text{ on } [t_k, t_{k+1}) \tag{5.3}$$
$$v_h^*(t) = \dot{Q}^{k+1} \text{ on } [t_k, t_{k+1}) \tag{5.4}$$
$$v_{*h}(t) = \dot{Q}^k \text{ on } [t_k, t_{k+1}) \tag{5.5}$$
$$F_h(t) = F(t_k, Q^k, \dot{Q}^k) \text{ on } [t_k, t_{k+1}). \tag{5.6}$$

It is easy to verify that

$$\dot{q_h}(t) = \dot{Q}^k = v_{*h}(t) \text{ on } (t_k, t_{k+1}) \tag{5.7}$$
$$\dot{v_h}(t) = \frac{\dot{Q}^{k+1} - \dot{Q}^k}{h_k} \text{ on } (t_k, t_{k+1}), \tag{5.8}$$

and

$$M(q_{*h}(t))\dot{v_h}(t) + C(q_{*h}(t), \dot{q_h}(t))v_h^*(t) + \nabla\mathcal{V}(q_{*h}(t)) + F_h(t) \in -\partial\Phi(v_h^*(t)) \ a.e. \text{ on } [0, T]. \tag{5.9}$$

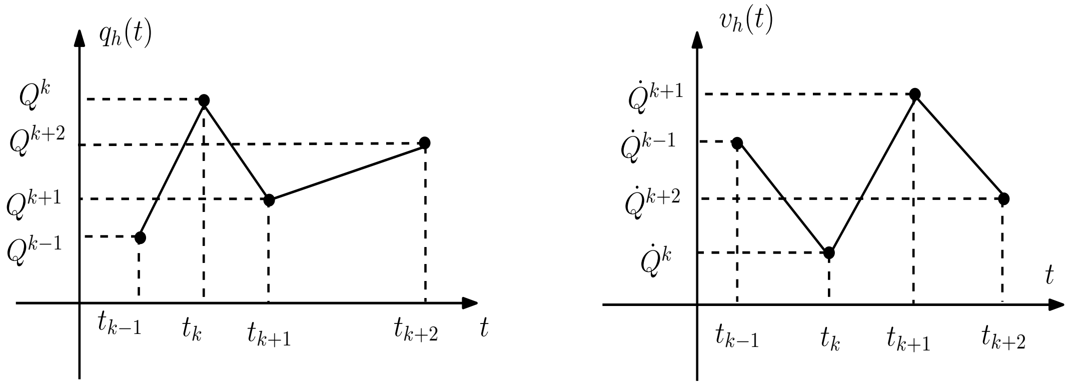

FIG. 5.1. *Piecewise linear approximation of position and velocity functions.*

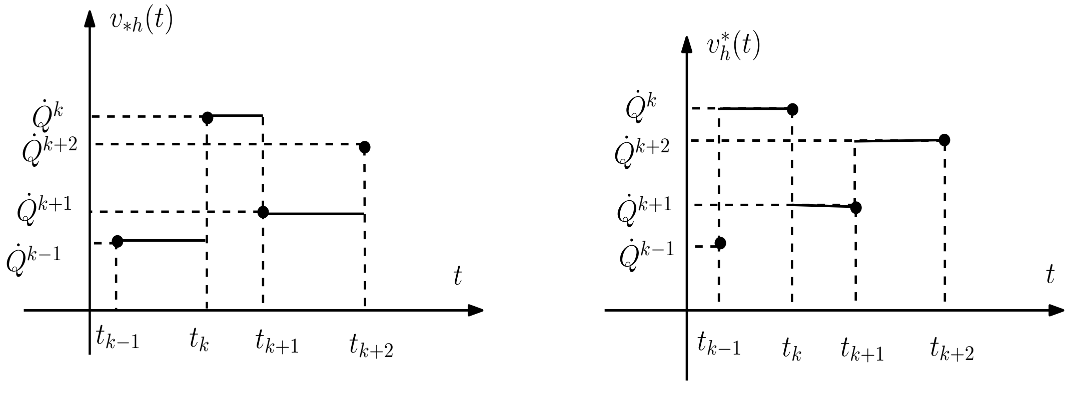

FIG. 5.2. *Step approximation of velocity functions from below (v_{*h}) and from above (v_h^*).*

Let us denote

$$C_1 := \sqrt{\frac{2E_0}{k_1}} \tag{5.10}$$

$$C_2 := \|Q^0\| + C_1 T, \tag{5.11}$$

where $E_0 = \frac{1}{2}\langle M(Q^0)\dot{Q}^0, \dot{Q}^0\rangle$. Note that C_1, C_2 are not dependent on h.

LEMMA 5.1. *Let Assumptions* 3.1, 3.2, 3.3 *and* 4.1 *hold. There exists a constant* M_1 *which is not dependent on* h *such that*

$$\|q_h\|_{\mathcal{C}(0,T;\mathbb{R}^n)} + \|q_{*h}\|_{L^\infty(0,T;\mathbb{R}^n)} + \|v_h\|_{\mathcal{C}(0,T;\mathbb{R}^n)} + \|\dot{q_h}\|_{L^\infty(0,T;\mathbb{R}^n)} + \|v_h^*\|_{L^\infty(0,T;\mathbb{R}^n)} + \|\dot{v}_h\|_{L^\infty(0,T;\mathbb{R}^n)} \le M_1. \tag{5.12}$$

Proof. Indeed, from (4.15), *it follows that* $\|\dot{Q}^k\| \le \sqrt{\frac{2E_0}{k_1}} = C_1$ *for all* $k = 1, 2...$ *Note that* $\dot{q_h}(t) = \dot{Q}^k$, *for all* $t \in (t_k, t_{k+1})$. *Hence*

$$\|v_h\|_{\mathcal{C}(0,T;\mathbb{R}^n)} + \|\dot{q_h}\|_{L^\infty(0,T;\mathbb{R}^n)} + \|v_h^*\|_{L^\infty(0,T;\mathbb{R}^n)} \le 3C_1. \tag{5.13}$$

Furthermore, for each integer $k > 0$, $\|Q^k\| \le \|Q^{k-1}\| + h_{k-1}\|\dot{Q}^{k-1}\| \le \|Q^{k-1}\| + h_{k-1}C_1 \le \ldots \le \|Q^0\| + (h_{k-1} + \ldots + h_0)C_1 \le \|Q^0\| + C_1 T = C_2$. *Therefore*

$$\|q_h\|_{\mathcal{C}(0,T;\mathbb{R}^n)} + \|q_{*h}\|_{L^\infty(0,T;\mathbb{R}^n)} \le 2C_2. \tag{5.14}$$

We have proved that the sequences $(\|\dot{Q}^k\|), (\|Q^k\|)$ *are bounded by* C_1 *and* C_2, *respectively. Using the boundedness of* $F(\cdot)$, *the boundedness on bounded sets of* $\partial\Phi(\cdot)$ *and Assumptions* 3.1, 3.2 $(\mathcal{C}_1)$, *from the differential inclusion* (0.2), *there exists a constant* C_3 *which is independent of* h *such that for all positive integers* k

$$\|\frac{\dot{Q}^{k+1} - \dot{Q}^k}{h_k}\| \le C_3. \tag{5.15}$$

It means that (see (5.8))

$$\|\dot{v}_h\|_{L^\infty(0,T;\mathbb{R}^n)} \le C_3. \tag{5.16}$$

Thus we can find a constant M_1 *which is not dependent on* h *such that* (5.12) *holds.* □

LEMMA 5.2. *Let Assumptions* 3.1, 3.2, 3.3 *and* 4.1 *hold. For all* $h > 0$, *there exists a constant* M_2 *which is not dependent on* h *such that*

$$\|q_h - q_{*h}\|_{L^2(0,T;\mathbb{R}^n)} + \|\dot{q_h} - v_h\|_{L^2(0,T;\mathbb{R}^n)} + \|v_h - v_h^*\|_{L^2(0,T;\mathbb{R}^n)} \le M_2 h. \tag{5.17}$$

Proof. Let us consider on the interval (t_k, t_{k+1}) for any positive integer k

$$q_h(t) - q_{*h}(t) = Q^k + \dot{Q}^k(t - t_k) - Q^k = \dot{Q}^k(t - t_k) \tag{5.18}$$

which implies that

$$\int_{t_k}^{t_{k+1}} \|q_h(t) - q_{*h}(t)\|^2 dt \le C_1^2 \int_{t_k}^{t_{k+1}} (t - t_k)^2 dt \le \frac{C_1^2}{3}(t_{k+1} - t_k)^3 \le \frac{C_1^2 h^2}{3}(t_{k+1} - t_k). \tag{5.19}$$

Hence

$$\int_0^T \|q_h(t) - q_{*h}(t)\|^2 dt \leq \frac{C_1^2 T h^2}{3}, \tag{5.20}$$

and thus $\|q_h - q_{*h}\|_{L^2(0,T;\mathbb{R}^n)} \leq \sqrt{\frac{T}{3}} C_1 h$. Similarly, one has on the interval (t_k, t_{k+1})

$$v_h(t) - \dot{q_h}(t) = \dot{Q}^k + \frac{\dot{Q}^{k+1} - \dot{Q}^k}{h_k}(t - t_k) - \dot{Q}^k = \frac{\dot{Q}^{k+1} - \dot{Q}^k}{h_k}(t - t_k) \tag{5.21}$$

$$v_h(t) - v_h^*(t) = \dot{Q}^{k+1} + \frac{\dot{Q}^{k+1} - \dot{Q}^k}{h_k}(t - t_{k+1}) - \dot{Q}^{k+1} = \frac{\dot{Q}^{k+1} - \dot{Q}^k}{h_k}(t - t_{k+1}) \tag{5.22}$$

and

$$\|v_h - \dot{q_h}\|_{L^2(0,T;\mathbb{R}^n)} \leq \sqrt{\frac{T}{3}} C_3 h \tag{5.23}$$

$$\|v_h - v_h^*\|_{L^2(0,T;\mathbb{R}^n)} \leq \sqrt{\frac{T}{3}} C_3 h, \tag{5.24}$$

due to (5.15). Then the conclusion follows. □

Lemma 5.3. *Let Assumptions* 3.1, 3.2, 3.3 *and* 4.1 *hold. In addition, suppose that* Assumption 5.1.

$$\forall R \geq 0, \;\; \Gamma(R) = \sup\left\{ \left\| \frac{\partial F}{\partial t}(\cdot, q, v) \right\|_{L^2(0,T;\mathbb{R}^n)} : \;\; \|q\|_{L^2(0,T;\mathbb{R}^n)}, \|v\|_{L^2(0,T;\mathbb{R}^n)} \leq R \right\} < +\infty. \tag{5.25}$$

Then for all $h > 0$, there exists a constant M_3 which is not dependent on h such that

$$\|F(\cdot, q_{*h}, \dot{q_h}) - F_h\|_{L^2(0,T;\mathbb{R}^n)} \leq M_3 h. \tag{5.26}$$

Proof. Let us consider on the interval (t_k, t_{k+1})

$$F(t, q_{*h}(t), \dot{q_h}(t)) - F_h(t) = F(t, Q^k, \dot{Q}^k) - F(t_k, Q^k, \dot{Q}^k) = \int_{t_k}^t \frac{\partial F}{\partial t}(s, Q^k, \dot{Q}^k) ds. \tag{5.27}$$

Then the proof is similar to the proof of Lemma 2.4 in [10]. □

In the following theorem, we show that convergent subsequence of (q_h, v_h) can be extracted such that its limit function is a solution of the continuous system (0.1).

Theorem 5.1. *Let Assumptions* 3.1, 3.2, 3.3, 4.1 *and* 5.1 *hold. Then for any given initial conditions, we can find a subsequence of (q_h, v_h) (defined in* (5.1), (5.2)*) converging to a solution of the continuous system* (0.1).

Proof. The Lemma 5.1 ensures that the sequences of continuous functions (q_h), (v_h) are uniformly bounded in $\mathcal{C}(0, T; \mathbb{R}^n)$ with the supremum norm and equicontinuous (since the sequences $(\dot{q_h})$, $(\dot{v}_h)$ are uniformly bounded in $L^\infty(0, T; \mathbb{R}^n)$). Then, by using the Arzelà-Ascoli Theorem, there exist cluster points $q, v \in \mathcal{C}(0,T;\mathbb{R}^n)$ and subsequences of them, still denoted by (q_h), (v_h) such that

$$q_h \to q \quad \text{in} \quad \mathcal{C}(0,T;\mathbb{R}^n) \tag{5.28}$$

$$v_h \to v \quad \text{in} \quad \mathcal{C}(0,T;\mathbb{R}^n). \tag{5.29}$$

In particular $v_h \to v$ in $L^2(0,T;\mathbb{R}^n)$. On the other hand, the Lemma 5.2 implies that $\|\dot{q_h} - v_h\|_{L^2(0,T;\mathbb{R}^n)} \to 0$. Therefore $\dot{q_h} \to v$ in $L^2(0,T;\mathbb{R}^n)$ due to the triangle inequality. We have $\dot{q_h} \to \dot{q}$ and $\dot{q_h} \to v$ in the sense of distribution on $(0,T)$. Identifying both limits, one infers that $\dot{q} = v$ in $L^2(0,T;\mathbb{R}^n)$. Since v is continuous on $(0,T)$, we deduce that the function $q(t) = q(0) + \int_0^t \dot{q}(s)ds = q(0) + \int_0^t v(s)ds$ is differentiable, or equivalently, $q \in \mathcal{C}^1(0,T;\mathbb{R}^n)$ (see for example, Theorem 8.2 in [14]). The Lemma 5.1 also implies that the sequence $(\dot{v_h})$ is bounded in $L^\infty(0,T;\mathbb{R}^n)$. By using Banach-Alaoglu Theorem, we can find a function $w \in L^\infty(0,T;\mathbb{R}^n)$ and a subsequence of $(\dot{v_h})$, still denoted by $(\dot{v_h})$, such that

$$\dot{v_h} \to w \quad \text{for the topology} \quad \sigma(L^\infty(0,T;\mathbb{R}^n), L^1(0,T;\mathbb{R}^n)). \tag{5.30}$$

One has $\dot{v_h} \to \dot{v} = \ddot{q}$ and $\dot{v_h} \to w$ in the sense of distribution on $(0,T)$. Identifying both limits, we obtain that $\ddot{q} \in L^\infty(0,T;\mathbb{R}^n)$, or equivalently, $q \in \mathcal{W}^{2,\infty}(0,T;\mathbb{R}^n)$. In the next part, we will prove that $q(\cdot)$ is a solution of (0.1).
Indeed, it is easy to verify that $q(\cdot)$ satisfies the initial condition. Note that $A(\cdot) := \partial\Phi(\cdot)$ is a maximal monotone operator from $\mathbb{R}^n$ into the subsets $\mathbb{R}^n$. Let us define the new operator $\mathcal{A}(\cdot)$ from $L^2(0,T;\mathbb{R}^n)$ into the subsets $L^2(0,T;\mathbb{R}^n)$ by

$$f \in \mathcal{A}(u) \Longleftrightarrow f(t) \in A(u(t)) \; a.e. \tag{5.31}$$

It is known that $\mathcal{A}(\cdot)$ is also maximal monotone, see [13]. From Lemma 5.2, $(\mathcal{H}_F)$ and (5.28), (5.29), (5.30) one obtains that

$$q_{*h} \to q \;\text{ strongly in }\; L^2(0,T;\mathbb{R}^n) \tag{5.32}$$

$$v_h^*, \dot{q_h} \to \dot{q} \;\text{ strongly in }\; L^2(0,T;\mathbb{R}^n) \tag{5.33}$$

$$F_h \to F(\cdot, q, \dot{q}) \;\text{ strongly in }\; L^2(0,T;\mathbb{R}^n) \tag{5.34}$$

$$\dot{v_h} \to \ddot{q} \;\text{ weakly in }\; L^2(0,T;\mathbb{R}^n). \tag{5.35}$$

Let us prove that

$$M(q_{*h})\dot{v_h} \to M(q)\ddot{q} \;\text{ weakly in }\; L^2(0,T;\mathbb{R}^n). \tag{5.36}$$

Given $\varphi \in L^2(0,T;\mathbb{R}^n)$, note that $M(q)\varphi$ also belongs to $L^2(0,T;\mathbb{R}^n)$. Then

$$\begin{aligned}
&|\int_0^T \langle M(q_{*h}(t))\dot{v_h}(t) - M(q(t))\ddot{q}(t), \varphi(t)\rangle dt| \\
&\le |\int_0^T \langle [M(q_{*h}(t)) - M(q(t))]\dot{v_h}(t), \varphi(t)\rangle d| + |\int_0^T \langle M(q(t))(\dot{v_h}(t) - \ddot{q}(t)), \varphi(t)\rangle dt| \\
&\le k_4 \int_0^T \|q_{*h}(t) - q(t)\|\|\dot{v_h}(t)\|\|\varphi(t)\|dt + |\int_0^T \langle (\dot{v_h}(t) - \ddot{q}(t)), M(q(t))\varphi(t)\rangle dt| \to 0
\end{aligned}$$

for some constant $k_4 > 0$ by using Lemma 3.1, the boundedness of $q_{*h}, q, \dot{v_h}$, and since $\dot{v_h}$ converges weakly to $\ddot{q}$ while q_{*h} converges to q strongly in $L^2(0,T;\mathbb{R}^n)$. Therefore, one obtains

$$M(q_{*h})\dot{v_h} + C(q_{*h}, \dot{q_h})v_h^* + \nabla\mathcal{V}(q_{*h}) + F_h \to M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla\mathcal{V}(q) + F(t,q,\dot{q})$$

weakly in $L^2(0,T;\mathbb{R}^n)$. From (5.9), (5.33) and the closed graph property of the maximal monotone operator $\mathcal{A}(\cdot)$, we infer that

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla\mathcal{V}(q) + F(t,q,\dot{q}) \in \mathcal{A}(\dot{q}). \tag{5.37}$$

It means that $(q,\dot{q})$ is a solution of (0.1). □

In the following part, we show the convergence order of the scheme under the following assumption, which is about the local hypo-monotonicity of the operator $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \mapsto M^{-1}(x)\partial\Phi(y)$, for any $x \in \mathbb{R}^n$. Note that if this assumption holds, then it is easy to verify that (0.1) has at most one solution for given initial condition (see Theorem 5.1 in [4]). In particular, the assumption is satisfied for the cases in Proposition 5.1 and Proposition 5.2 in [4].

Assumption 5.2. *Given $K > 0$, there exists a constant $c > 0$ such that for all $x, y_1, y_2 \in \mathbb{B}_K$ and $y_1^* \in \partial\Phi(y_1), y_2^* \in \partial\Phi(y_2)$*

$$\langle M^{-1}(x_1)y_1^* - M^{-1}(x_2)y_2^*, y_1 - y_2\rangle \geq -c(\|x_1 - x_2\|^2 + \|y_1 - y_2\|^2). \tag{5.38}$$

Theorem 5.2. *Let Assumptions 3.1, 3.2, 3.3, 4.1, 5.1 and 5.2 hold. Then for any given initial conditions, the piecewise linear approximate function (q_h, v_h) converges to the unique solution of the continuous system (0.1) with order of convergence 1/2.*

Proof. For all $h, k > 0$ and accepting a usual abuse of notations, we get

$$-[M(q_{*h})\dot{v_h} + C(q_{*h}, \dot{q_h})v_h^* + \nabla\mathcal{V}(q_{*h}) + F_h] \in \partial\Phi(v_h^*) \tag{5.39}$$
$$-[M(q_{*k})\dot{v_k} + C(q_{*k}, \dot{q_k})v_k^* + \nabla\mathcal{V}(q_{*k}) + F_k] \in \partial\Phi(v_k^*). \tag{5.40}$$

By using Assumption 5.2, one infers

$$\begin{aligned}&\langle \dot{v_h} - \dot{v_k} + M^{-1}(q_{*h})[C(q_{*h}, \dot{q_h})v_h^* + \nabla\mathcal{V}(q_{*h}) + F_h] - M^{-1}(q_{*k})[C(q_{*k}, \dot{q_k})v_k^* \\ &+\nabla\mathcal{V}(q_{*k}) + F_k], v_h^* - v_k^*\rangle \leq c(\|q_{*h} - q_{*k}\|^2 + \|v_h^* - v_k^*\|^2).\end{aligned} \tag{5.41}$$

Let us integrate both sides of (5.41) on $[0,t]$ $(t \in (0,T))$ and estimate all the terms. We begin with the first term by using Cauchy-Schwarz inequality

$$\begin{aligned}\int_0^t \langle \dot{v_h} - \dot{v_k}, v_h^* - v_k^*\rangle &= \frac{1}{2}\|v_h(t) - v_k(t)\|^2 + \int_0^t \langle \dot{v_h} - \dot{v_k}, v_h^* - v_h + v_k - v_k^*\rangle \\ &\geq \frac{1}{2}\|v_h(t) - v_k(t)\|^2 - (\|\dot{v_h}\|_{L^2(0,T;\mathbb{R}^n)} + \|\dot{v_k}\|_{L^2(0,T;\mathbb{R}^n)})(\|v_h^* - v_h\|_{L^2(0,T;\mathbb{R}^n)} \\ &\quad +\|v_k - v_k^*\|_{L^2(0,T;\mathbb{R}^n)}) \\ &\geq \frac{1}{2}\|v_h(t) - v_k(t)\|^2 - 2\sqrt{T}M_1M_2(h+k),\end{aligned} \tag{5.42}$$

where M_1, M_2 are defined in Lemma 5.1 and Lemma 5.2. Let us now deal with the remaining terms in (5.41). From the Assumptions 3.2 $(\mathcal{C}_1)$, $(\mathcal{C}_3)$, 3.3 $(\mathcal{H}_\mathcal{V})$, $(\mathcal{H}_F)$, Lemma 3.2, Lemma 3.3 and Lemma 5.1, it is not difficult to find some constants $c_1, c_2, c_3 > 0$ such that

$$\|M^{-1}(q_{*h})C(q_{*h},\dot{q_h})v_h^*-M^{-1}(q_{*k})C(q_{*k},\dot{q}_k)v_k^*\| \le c_1(\|q_{*h}-q_{*k}\|+\|\dot{q_h}-\dot{q}_k\|+\|v_h^*-v_k^*\|) \tag{5.43}$$

$$\|M^{-1}(q_{*h})\nabla\mathcal{V}(q_{*h}) - M^{-1}(q_{*k})\nabla\mathcal{V}(q_{*k})\| \le c_2\|q_{*h} - q_{*k}\| \tag{5.44}$$

and

$$\|M^{-1}(q_{*h})F(\cdot,q_{*h},\dot{q_h})-M^{-1}(q_{*k})F(\cdot,q_{*k},\dot{q}_k)\| \le c_3(\|q_{*h}-q_{*k}\|+\|\dot{q_h}-\dot{q}_k\|). \tag{5.45}$$

Indeed, let us prove the first inequality, the two remaining inequalities are proved similarly. By using the triangle inequality, one infers

$$\begin{aligned}
&\|M^{-1}(q_{*h})C(q_{*h},\dot{q_h})v_h^* - M^{-1}(q_{*k})C(q_{*k},\dot{q}_k)v_k^*\| \\
&\le \|[M^{-1}(q_{*h}) - M^{-1}(q_{*k})]C(q_{*h},\dot{q_h})v_h^*\| + \|M^{-1}(q_{*k})[C(q_{*h},\dot{q_h})v_h^* - C(q_{*k},\dot{q}_k)v_k^*]\| \\
&\le m_1\|q_{*h} - q_{*k}\| + m_2(\|q_{*h} - q_{*k}\| + \|\dot{q_h} - \dot{q}_k\| + \|v_h^* - v_k^*\|) \\
&\le c_1(\|q_{*h} - q_{*k}\| + \|\dot{q_h} - \dot{q}_k\| + \|v_h^* - v_k^*\|)
\end{aligned} \tag{5.46}$$

for some constants $m_1, m_2 > 0$ and $c_1 := m_1+m_2$ due to Assumption 3.2 ($\mathcal{C}_1$), Lemma 3.2, Lemma 5.1, Lemma 3.3 and Assumptions ($\mathcal{C}_3$).

From the inequality (5.45) and the triangle inequality, we have

$$\begin{aligned}
&\|M^{-1}(q_{*h})F_h - M^{-1}(q_{*k})F_k\| \\
&\le \|M^{-1}(q_{*h})[F_h - F(\cdot,q_{*h},\dot{q_h})]\| + \|M^{-1}(q_{*h})F(\cdot,q_{*h},\dot{q_h}) - M^{-1}(q_{*k})F(\cdot,q_{*k},\dot{q}_k)\| \\
&\quad +\|M^{-1}(q_{*k})[F_k - F(\cdot,q_{*k},\dot{q}_k)]\| \\
&\le k_6(\|F_h - F(\cdot,q_{*h},\dot{q_h})\| + \|F_k - F(\cdot,q_{*k},\dot{q}_k)\|) + c_3(\|q_{*h} - q_{*k}\| + \|\dot{q_h} - \dot{q}_k\|)
\end{aligned} \tag{5.47}$$

where k_6 is defined in Lemma 3.2. By using the inequality $ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$ and Lemma 5.3, one gets

$$\begin{aligned}
&\int_0^t (\|F_h - F(\cdot,q_{*h},\dot{q_h})\| + \|F_k - F(\cdot,q_{*k},\dot{q}_k)\|)\|v_h^* - v_k^*\|ds \\
&\le \frac{1}{2}\int_0^T (\|F_h - F(\cdot,q_{*h},\dot{q_h})\|^2 + \|F_k - F(\cdot,q_{*k},\dot{q}_k)\|^2)ds + \int_0^t \|v_h^* - v_k^*\|^2 ds \\
&\le \frac{1}{2}M_3^2(h+k)^2 + \int_0^t \|v_h^* - v_k^*\|^2 ds,
\end{aligned} \tag{5.48}$$

where M_3 is defined in Lemma 5.3 . Let us recapitulate the inequalities in (5.42)-(5.48):

$$\int_0^t \langle \dot{v_h} - \dot{v_k}, v_h^* - v_k^*\rangle ds \ge \frac{1}{2}\|v_h(t) - v_k(t)\|^2 - 2\sqrt{T}M_1M_2(h+k) \tag{5.49}$$

$$\begin{aligned}
&\int_0^t \langle M^{-1}(q_{*h})C(q_{*h},\dot{q_h})v_h^* - M^{-1}(q_{*k})C(q_{*k},\dot{q}_k)v_k^*, v_h^* - v_k^*\rangle ds \ge \\
&\qquad -c_1\int_0^t (\|q_{*h} - q_{*k}\| + \|\dot{q_h} - \dot{q}_k\| + \|v_h^* - v_k^*\|)\|v_h^* - v_k^*\|ds
\end{aligned} \tag{5.50}$$

$$\int_0^t \langle M^{-1}(q_{*h})\nabla\mathcal{V}(q_{*h}) - M^{-1}(q_{*k})\nabla\mathcal{V}(q_{*k}), v_h^* - v_k^*\rangle ds \geq -c_2 \int_0^t \|q_{*h} - q_{*k}\|\|v_h^* - v_k^*\| ds \tag{5.51}$$

$$\begin{aligned}\int_0^t \langle M^{-1}(q_{*h})F_h - M^{-1}(q_{*k})F_k, v_h^* - v_k^*\rangle ds &\geq -k_6 \int_0^t (\|F_h - F(\cdot, q_{*h}, \dot{q_h})\| \\ +\|F_k - F(\cdot, q_{*k}, \dot{q_k})\|)\|v_h^* - v_k^*\| ds &- c_3 \int_0^t (\|q_{*h} - q_{*k}\| + \|\dot{q_h} - \dot{q_k}\|)\|v_h^* - v_k^*\| ds \\ \geq -\frac{1}{2}k_6 M_3^2(h+k)^2 - k_6 \int_0^t \|v_h^* - v_k^*\|^2 ds &- c_3 \int_0^t (\|q_{*h} - q_{*k}\| + \|\dot{q_h} - \dot{q_k}\|)\|v_h^* - v_k^*\| ds.\end{aligned} \tag{5.52}$$

Let $c_4 := c_1 + c_2 + c_3$. Taking the sum of (5.49)-(5.52) side by side and combining with (5.41), we obtain

$$\begin{aligned}&\frac{1}{2}\|v_h(t) - v_k(t)\|^2 - 2\sqrt{T}M_1M_2(h+k) - \frac{1}{2}k_6M_3^2(h+k)^2 - c_4 \int_0^t (\|q_{*h} - q_{*k}\| + \|\dot{q_h} - \dot{q_k}\| \\ &+ \|v_h^* - v_k^*\|)\|v_h^* - v_k^*\| ds - k_6 \int_0^t \|v_h^* - v_k^*\|^2 ds \leq \int_0^t \langle \dot{v_h} - \dot{v_k} + M^{-1}(q_{*h})[C(q_{*h}, \dot{q_h})v_h^* \\ &+ \nabla\mathcal{V}(q_{*h}) + F_h] - M^{-1}(q_{*k})[C(q_{*k}, \dot{q_k})v_k^* + \nabla\mathcal{V}(q_{*k}) + F_k], v_h^* - v_k^*\rangle ds \\ &\leq c \int_0^t (\|q_{*h} - q_{*k}\|^2 + \|v_h^* - v_k^*\|^2) ds.\end{aligned} \tag{5.53}$$

Thus, one infers

$$\begin{aligned}\frac{1}{2}\|v_h(t) - v_k(t)\|^2 &\leq 2\sqrt{T}M_1M_2(h+k) + \frac{1}{2}k_6M_3^2(h+k)^2 + c_4 \int_0^t (\|q_{*h} - q_{*k}\| + \|\dot{q_h} - \dot{q_k}\| \\ &+ \|v_h^* - v_k^*\|)\|v_h^* - v_k^*\| ds + \int_0^t (c\|q_{*h} - q_{*k}\|^2 + (c + k_6)\|v_h^* - v_k^*\|^2) ds \\ &\leq M_4(h+k) + \frac{c_4}{2} \int_0^t (\|q_{*h} - q_{*k}\|^2 + \|\dot{q_h} - \dot{q_k}\|^2) ds \\ &+ 2c_4 \int_0^t \|v_h^* - v_k^*\|^2 ds + \int_0^t (c\|q_{*h} - q_{*k}\|^2 + (c + k_6)\|v_h^* - v_k^*\|^2) ds \\ &\leq M_4(h+k) + c_5 \int_0^t (\|q_{*h} - q_{*k}\|^2 + \|\dot{q_h} - \dot{q_k}\|^2 + \|v_h^* - v_k^*\|^2) ds \end{aligned} \tag{5.54}$$

where $c_5 := c + 2c_4 + k_6$ and $M_4 := 2\sqrt{T}M_1M_2 + \frac{1}{2}k_6M_3^2$ (h and k are chosen small enough such that $h + k \leq 1$).

The Lemma 5.2 implies

$$\begin{aligned}\int_0^t \|q_{*h}-q_{*k}\|^2 ds &\le \int_0^t (\|q_{*h}-q_h\|+\|q_h-q_k\|+\|q_k-q_{*k}\|)^2 ds\\ &\le 3\int_0^t (\|q_h-q_k\|^2+\|q_{*h}-q_h\|^2+\|q_k-q_{*k}\|^2)ds\\ &\le 3\int_0^t \|q_h-q_k\|^2 ds + 3\int_0^T (\|q_{*h}-q_h\|^2+\|q_k-q_{*k}\|^2)ds\\ &\le 3\int_0^t \|q_h-q_k\|^2 ds + 3M_2^2(h+k)^2. \qquad (5.55)\end{aligned}$$

Similarly, we have

$$\int_0^t \|\dot{q_h}-\dot{q}_k\|^2 ds \le 3\int_0^t \|v_h-v_k\|^2 ds + 3M_2^2(h+k)^2 \qquad (5.56)$$

$$\int_0^t \|v_h^*-v_k^*\|^2 ds \le 3\int_0^t \|v_h-v_k\|^2 ds + 3M_2^2(h+k)^2. \qquad (5.57)$$

On the other hand

$$\frac{1}{2}\|q_h(t)-q_k(t)\|^2 = \int_0^t \langle \dot{q_h}-\dot{q}_k, q_h-q_k\rangle dt \le \frac{1}{2}\int_0^t (\|\dot{q_h}-\dot{q}_k\|^2+\|q_h-q_k\|^2)ds. \quad (5.58)$$

From (5.54)-(5.58), one can find some constants $M_5>0$ and $c_6>0$ such that (with $h+k\le 1$)

$$\|q_h(t)-q_k(t)\|^2+\|v_h(t)-v_k(t)\|^2 \le M_5(h+k)+c_6\int_0^t (\|q_h(s)-q_k(s)\|^2+\|v_h(s)-v_k(s)\|^2)ds.$$

By using the Gronwall's inequality with the function $\Gamma(\cdot):=\|q_h(\cdot)-q_k(\cdot)\|^2+\|v_h(\cdot)-v_k(\cdot)\|^2$, one deduces

$$\Gamma(t)=\|q_h(t)-q_k(t)\|^2+\|v_h(t)-v_k(t)\|^2 \le M_6(h+k) \qquad (5.59)$$

where $M_6=M_5(h+k)e^{c_6 T}$. Let $k\to 0$, we obtain

$$\|q_h(t)-q(t)\|^2+\|v_h(t)-v(t)\|^2 \le M_6 h. \qquad (5.60)$$

It means that the piecewise linear approximate function converges to the unique solution of the continuous system (0.1) with order $1/2$. □

REMARK 5.1. *If the matrix $M(q)$ is constant, by using a transformation of variables, we can reduce our systems to first order differential inclusions where the right-hand side is the subdifferential of a convex function. Then the convergence order is* 1, *see [9].*

6. Numerical Simulations. Let us consider the following dynamical model of a robot manipulator (see [11]) without perturbation ($F\equiv 0$), which is only for illustrative purpose:

$$M(q)\ddot{q}+C(q,\dot{q})\dot{q}+\nabla\mathcal{V}(q)\in -\partial\Phi(\dot{q}), \qquad (6.1)$$

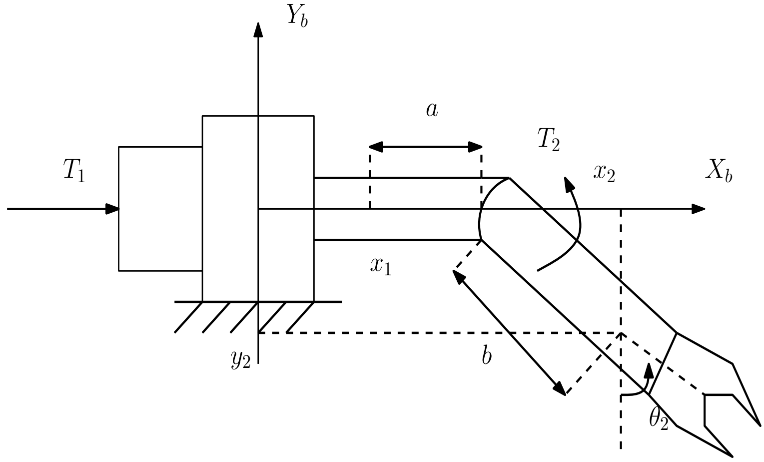

Fig. 6.1. *A robot manipulator.*

where the state $q = \begin{pmatrix} x_1 \\ \theta_2 \end{pmatrix}$ and

$$M(q) = \begin{pmatrix} m_1 + m_2 & m_2 b \cos\theta_2 \\ m_2 b \cos\theta_2 & I + m_2 b^2 \end{pmatrix}, \quad C(q, \dot{q}) = \begin{pmatrix} 0 & -m_2 b \dot{\theta}_2 \sin\theta_2 \\ 0 & 0 \end{pmatrix},$$

$$\nabla \mathcal{V}(q) = \begin{pmatrix} 0 \\ b m_2 g \sin\theta_2 \end{pmatrix}, \quad \partial\Phi(\dot{q}) = -\begin{pmatrix} T_1 \\ T_2 \end{pmatrix},$$

for some positive constants m_1, m_2, b, I and g. If we consider the forces T_1, T_2 with the following form

$$T_1 = -(2\eta_1 \dot{x}_1 + \alpha_1 \mathrm{Sign}(\dot{x}_1)) \ , \ T_2 = -(2\eta_2 \dot{\theta}_2 + \alpha_2 \mathrm{Sign}(\dot{\theta}_2))$$

for some positive constants $\eta_1, \eta_2, \alpha_1, \alpha_2$, then for $y = (y_1 \ y_2)^T \in \mathbb{R}^2$

$$\Phi(y) = \eta_1 y_1^2 + \alpha_1 |y_1| + \eta_2 y_2^2 + \alpha_2 |y_2| \geq \min\{\eta_1, \eta_2\} \|y\|^2 + \min\{\alpha_1, \alpha_2\} \|y\|.$$

The control forces $T_1(\cdot)$ and $T_2(\cdot)$ correspond to velocity plus sliding-mode feedback controller, a control technique that is used in practice [23]. It is not difficult to show that Assumptions $3.1, 3.2, 3.3, 4.1$ and 5.1 are satisfied with suitable parameters, but not Assumption 5.2. This assumption, which is also a sufficient condition to obtain the uniqueness, holds for a narrow class of mechanical systems. In fact, the property of solution uniqueness of nonsmooth problems is not attained generally. However, it is important and interesting to simulate at least one solution of the continuous problems. For the sake of simplicity, we consider $\Phi(y) = \frac{1}{2}\|y\|^2 + |y_1| + |y_2|$, which implies $\partial\Phi(y) = y + \mathrm{Sign}(y)$ where

$$\mathrm{Sign}(y) = \begin{pmatrix} \mathrm{Sign}(y_1) \\ \mathrm{Sign}(y_2) \end{pmatrix}, \ \text{ for } \ m \in \mathbb{R}, \mathrm{Sign}(m) = \begin{cases} -1 & \text{if} \quad m < 0, \\ [-1, +1] & \text{if} \quad m = 0, \\ +1 & \text{if} \quad m > 0. \end{cases} \tag{6.2}$$

From the first line of (0.2), we have

$$M(Q^k)\frac{\dot{Q}^{k+1}-\dot{Q}^k}{h_k}+C(Q^k,\dot{Q}^k)\dot{Q}^{k+1}+\nabla\mathcal{V}(Q^k)\in-\dot{Q}^{k+1}-\mathrm{Sign}(\dot{Q}^{k+1}). \tag{6.3}$$

Let us denote

$$P\triangleq\frac{1}{h_k}M(Q^k)+C(Q^k,\dot{Q}^k)+I_2\ ;\ x\triangleq\nabla\mathcal{V}(Q^k)-\frac{1}{h_k}M(Q^k)\dot{Q}^k. \tag{6.4}$$

This leads to find a solution y of the following inclusion for given x

$$Py+x\in-\mathrm{Sign}(y)=\partial\phi(y), \tag{6.5}$$

where $\phi:\mathbb{R}^2\to\mathbb{R}$ is a convex function defined by

$$\phi(y)=|y_1|+|y_2|. \tag{6.6}$$

Note that at each step, the matrix P is positive definite (in fact $P\geq I_2$, see Theorem 4.1) but non-symmetric since the Coriolis matrix C is non-symmetric. The existence and uniqueness of y in (6.5) are assured by a well-known result for general function ϕ (proper, convex and lower semicontinuous), see for example [3]. Furthermore, the operator $(D+\partial\phi)^{-1}$ is single-valued and Lipschitz continuous. For the specific function $\phi(\cdot)$ defined in (6.6) and if the matrix P is symmetric positive definite then it is possible to write $(D+\partial\phi)^{-1}$ as a projection operator by some change of variable coordinates (see Lemma 6.1). In our case, P is non-symmetric. Denote the symmetric part and antisymmetric part of P by

$$P_s\triangleq\frac{P+P^T}{2};\ \ P_a\triangleq\frac{P-P^T}{2}. \tag{6.7}$$

Let $K\triangleq[-1,1]^2$ be the square containing the origin in $\mathbb{R}^2$ and $L\triangleq\{z\in\mathbb{R}^2:P_s^{1/2}z\in K\}=P_s^{-1/2}K$. The following result allows us to calculate numerically the velocity at each step.

LEMMA 6.1. *For given $x\in\mathbb{R}^2$, the inclusion* (6.5) *is equivalent to the equation*

$$[I+P_s^{-1}P_a]y+P_s^{-1}x=P_s^{-1/2}\mathrm{proj}[L;P_s^{-1/2}P_ay+P_s^{-1/2}x]. \tag{6.8}$$

Proof. Note that the inclusion

$$-Py-x\in\partial\phi(y) \tag{6.9}$$

is equivalent to

$$y\in\partial\phi^*(-Py-x)=\partial\Psi_K(-Py-x)=N_K(-Py-x)=-N_K(Py+x), \tag{6.10}$$

by using (2.2), (2.3) and the symmetry of the set K. Let $x'\triangleq P_ay+x$. From (6.10), we obtain

$$y\in-N_K(P_sy+x').$$

Let the vector z be such that $P_s^{1/2}z = P_s y + x'$, which implies $y = P_s^{-1/2}z - P_s^{-1}x'$. Then, we have

$$\begin{array}{lcl} P_s^{-1/2}z - P_s^{-1}x' \in -N_K(P_s^{1/2}z) & \Leftrightarrow & z - P_s^{-1/2}x' \in -P_s^{1/2}N_K(P_s^{1/2}z) \\ \Leftrightarrow z - P_s^{-1/2}x' \in -N_L(z) \ \text{ with } & & L = \{z \in \mathbb{R}^2 : P_s^{1/2}z \in K\} = P_s^{-1/2}K \\ \Leftrightarrow z = \text{proj}[L; P_s^{-1/2}x'] & \Leftrightarrow & y = P_s^{-1/2}\text{proj}[L; P_s^{-1/2}x'] - P_s^{-1}x' \\ \Leftrightarrow [I + P_s^{-1}P_a]y + P_s^{-1}x & = & P_s^{-1/2}\ \text{proj}[L;\ P_s^{-1/2}P_a y + P_s^{-1/2}x]. \end{array}$$

□

REMARK 6.1. *The equation* (6.8) *can be rewritten as* $\mathcal{F}(y) = 0$ *for some function* $\mathcal{F}(\cdot)$ *which is semismooth (it is not difficult to check). Then iterative solvers for semismooth functions may be used to solve it at each step [17]. In fact we can find explicitly the solution of the inclusion* (6.5) *in* $\mathbb{R}^2$ *but Lemma* 6.1 *permits us to compute numerically the velocity in general* R^n*, especially when the dimension is large, which is more practical.*

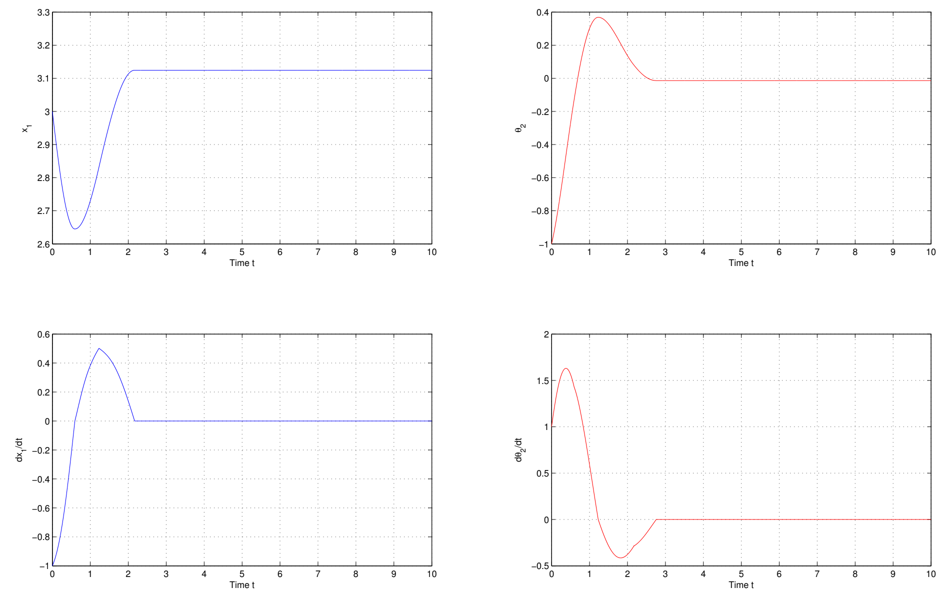

FIG. 6.2. *Simulation for the position and velocity of (6.1) with* $(x_1^0, \theta_2^0) = (3, -1)$ *and* $(\dot{x}_1^0, \dot{\theta}_2^0) = (-1, 1)$.

Now, let us simulate the system (6.1) with different initial conditions (Figures 6.2 and 6.3). Furthermore, we also consider the case with nonzero perturbation, for example with $F(t, x_1) = (sin(t)\ sin(x_1))^T$, with the same analysis (Figures 6.4 and 6.5). The finite time convergence and smooth stabilization at zero are clearly illustrated on all the figures. Another important point is that implicit methods allow to obtain the exact (at the machine precision) stabilization at the equilibrium, which is not the case with explicit methods. This type of behaviour for implicit methods is also studied

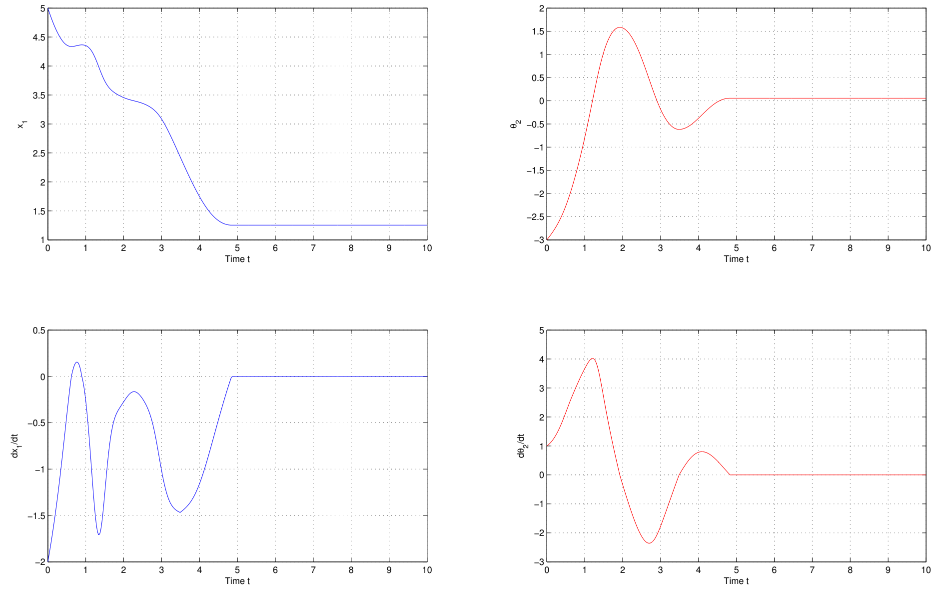

FIG. 6.3. *Simulation for the position and velocity of (6.1) with* $(x_1^0, \theta_2^0) = (5, -3)$ *and* $(\dot{x}_1^0, \dot{\theta}_2^0) = (-2, 1)$.

for similar but less general systems in [1, 2]. Furthermore, the example allows us to recover numerically the finite time convergence property studied in [4].

7. Conclusions. This paper contains the analysis of the time-discretization of the differential inclusions studied in [4] with an implicit Euler-like scheme. These differential inclusions stem from the robust, set-valued control of fully actuated Lagrangian systems. Applications are in the sliding mode control of mechanical systems. The convergence of the approximated positions and velocities is shown, as well as the order of the method. The proof is based on the use of a Lyapunov function, and of fundamental properties of Lagrangian systems. The convergence to the equilibrium, when it exists, is proved to occur in a finite number of steps. Numerical simulations illustrate the theoretical analysis.

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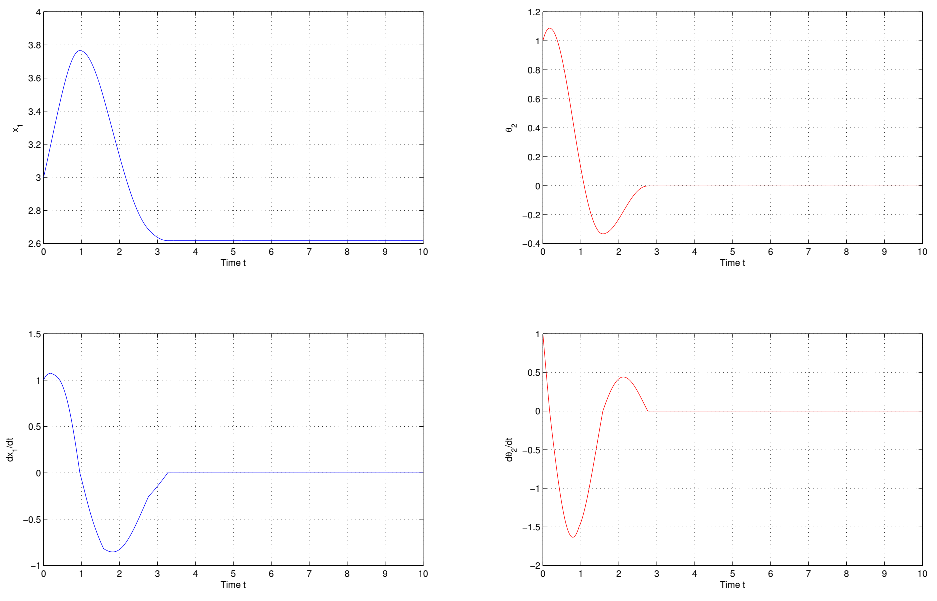

FIG. 6.4. *The system under perturbation with* $(x_1^0, \theta_2^0) = (3, 1)$ *and* $(\dot{x}_1^0, \dot{\theta}_2^0) = (1, 1)$.

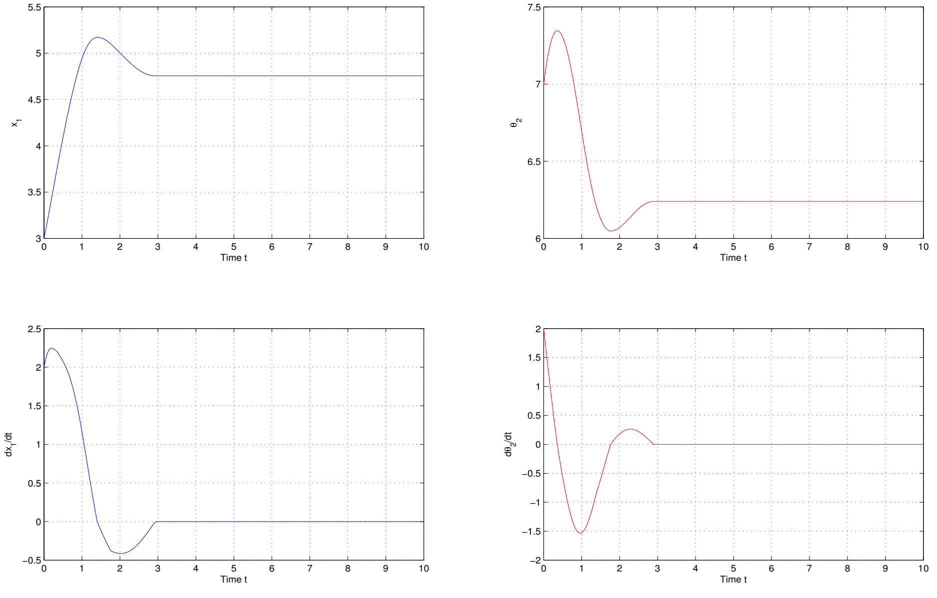

FIG. 6.5. *The system under perturbation with* $(x_1^0, \theta_2^0) = (3, 7)$ *and* $(\dot{x}_1^0, \dot{\theta}_2^0) = (2, 2)$.

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